# On a Class of Nonlocal SIR Models [*][†]

Li Guan,   Dong Li,   Ke Wang,   Kun Zhao

November 26, 2018

### Abstract

We revisit the classic Susceptible-Infected-Recovered (SIR) epidemic model and one of its recently developed nonlocal variations. We introduce several new approaches to derive exact analytical solutions in the classical situation and analyze the corresponding effective approximations in the nonlocal setting. An interesting new feature of the nonlocal models, compared with the classic SIR model, is the appearance of multiple peak solutions for the infected population. We provide several rigorous results on the existence and non-existence of peak solutions with sharp asymptotics.

## 1   Introduction

This paper is oriented around the classic SIR model in mathematical epidemiology and one of its recently developed nonlocal variations. The main purpose is to provide a systematic approach for finding exact analytical solutions to the models under appropriate initial conditions. The susceptible-infected-recovered (SIR) model has been a cornerstone in the study of the spreading mechanisms of infectious diseases for many decades since its initiation in the 1920s (Kermack and McKendrick 1927). The success of the SIR model has been a consequence of its intuitive simplicity, analytical tractability, and ability to predict the underlying mechanisms elucidating the spread of infectious diseases. In its simplest form, the classic SIR model is a system of nonlinear ordinary differential equations:

$$\begin{aligned}
\frac{dS}{dt} &= -\lambda \frac{I}{N} S, \\
\frac{dI}{dt} &= \lambda \frac{I}{N} S - \gamma I, \\
\frac{dR}{dt} &= \gamma I,
\end{aligned} \tag{1.1}$$

where $S, I, R$ and $N = S+R+I$ denote the susceptible, infected, recovered and total populations, respectively; $\lambda > 0$ is the total transmission rate, and $\gamma > 0$ is the average recovery rate. One should note that due to the conservation law $S + I + R \equiv N$ the degree of freedom in (1.2) is

in fact two instead of three. By a normalization of $\frac{S}{N} \to S$, $\frac{I}{N} \to I$, $\frac{R}{N} \to R$, the system (1.1) becomes

$$\frac{dS}{dt} = -\lambda I S,$$
$$\frac{dI}{dt} = \lambda I S - \gamma I, \tag{1.2}$$
$$\frac{dR}{dt} = \gamma I,$$

and the conservation law becomes $S + I + R \equiv 1$. Despite its simple looking structure, the nonlinear interaction between the susceptible and infected populations makes the investigation of some of the fundamental mathematical properties of the model a daunting task. In particular, the search of closed form (explicit) solutions to the model subject to appropriate initial conditions has been a significant challenge in the community of mathematical epidemiology, yet the resolution is still elusive. To the authors' knowledge, the implicit solution discovered by Harko, Lobo and Mak (2014) is the first exact analytical solution to the model since the 1920s. Although explicit solutions are still not available, the implicit solution can be charted in a systematic way such that it can be applied in various situations for practical purposes. An exact solution of a special case of the SIR model was presented earlier by Shabbir, Khan and Sadiq (arXiv:1012.5035. 2010, unpublished). The SIR model has also been investigated numerically in a number of works (Batiha and Batiha 2011; Biazar 2006; Rafei, Daniali and Ganji 2007; Rafei, Ganji and Daniali 2007). A stochastic epidemic-type model was recently studied by Williams, Mazilu and Mazilu (2012). Lyapunov functions for classic SIR and SIS epidemiological models were introduced by Korobeinikov and Wake (2002), and a new one later by O'Regan et al. (2010). For reaction-diffusion type models one can see recent (Li, Peng and Wang 2018; Lou and Zhao 2010, 2011) and the references therein for more detailed discussions.

## 1.1 Motivation and Goal

Now we would like to point out the facts that motivate the current work. Recently Guan, Hengartner and Hyman (personal communication) proposed an alternative version of the classic SIR model in which the deterministic exponential type harzard function is generalized to a class of cumulative distribution functions. The main idea is to replace the constant recovery rate in the classical situation by a more realistic variable recovery rate sampled from a general probability density distribution. Indeed, when the recovery rate is constant, upon replacing the nonlinear term in the second equation of (1.2) by $-S'(t)$, then formally solving the equation for $I(t)$, one gets

$$I(t) = \int_0^t e^{-\gamma(t-\tau)}[-S'(\tau)]d\tau + I_0 e^{-\gamma t},$$

where $' = \frac{d}{dt}$. One can then rewrite the exponential kernel function as $e^{-\gamma t} = 1 - (1 - e^{-\gamma t})$, and replace $1 - e^{-\gamma t}$ (note that this is exactly the cumulative density function of the exponential distribution) by a cumulative density function $G = G(t)$ supported in the positive axis $[0, \infty)$. This together with the differential equation for $S$ and the conservation of total population then

lead to the following nonlocal (normalized) SIR model:

$$\frac{dS}{dt} = -\lambda IS,$$

$$I(t) = \int_0^t [1 - G(t - \tau)] \cdot [-S'(\tau)] d\tau + I_0[1 - G(t)], \qquad (1.3)$$

$$R = 1 - I - S.$$

We observe that once the constant recovery rate is replaced by a probability density function, the approach of Harko, Lobo and Mak (2014), which is built on analyzing the usual Abel-type equations, becomes unaccessible for finding analytical solutions to the nonlocal model. Furthermore, even the construction and analysis of the corresponding approximate solutions seem nontrivial without deeper understanding of the nonlocal model. These are the major facts that motivate the current work.

The goal of this paper is to provide two alternative (more systematic) approaches for finding analytic and approximate solutions to the classic and nonlocal SIR models, (1.2) & (1.3), under appropriate initial conditions. The first approach (see Sections 2 and 4), is to regard the infected population, $I$, as a function of the susceptible one, $S$, instead of $t$, based on the observation that $S$ is a strictly decreasing function of $t$ (hence the inverse function of $S(t)$ exists). Then by solving a differential equation for $I$ in terms of $S$, one gets a closed form of $I$ in terms of $S$. In the nonlocal setting, one has to make certain approximations of the nonlocal kernel in order to derive a quantitative explicit expression. In the second approach (see Section 3 and esp. Section 5), we introduce a rescaled (nonlocal) time variable $\tau$ which is measured according to normalized instantaneous infected population, and analyze directly the susceptible population (along with other variables) by an approximation of the interaction kernel. All properties of the solutions are analyzed in the $\tau$-scale. We then translate these results into the original time scale $t$ by using the nonlocal map $t \to \tau$. Conceptually, this approach is rather clean since it corresponds to a nonlocal stretching of the time axis under which the dynamics of the model becomes much simpler (this is particularly transparent in the classic SIR case, see Section 3). On the other hand, an interesting new feature of the nonlocal model is that one can accommodate certain infected populations with several peaks (in the classic SIR case, one can have at most one, see Section 3). We give several rigorous constructions on the existence of such solutions having multiple peaks (with "controlled" centers) in Section 5.

The rest of this paper is organized as follows. In Sections 2 and 3, we revisit the classic SIR model and introduce the aforementioned analyses in the classic situation. These analyses are then generalized to the nonlocal SIR models in Sections 4 and 5. Some concluding remarks are gathered at the end of Section 5.

## Notation.

For any two positive quantities $X$ and $Y$, we shall use the notation $X \lesssim Y$ if $X \leq CY$ for some harmless positive constant $C$. We shall write $X \ll Y$ if $X \leq cY$ for some sufficiently small constant $c > 0$. The smallness of the constant $c$ is usually clear from the context.

## 2 Classic SIR Model

In this section, we revisit the classic SIR model which will motivate the nonlocal model in later sections. The classic (normalized) SIR model takes the form:

$$\frac{dS}{dt} = -\lambda IS,$$
$$\frac{dI}{dt} = \lambda IS - \gamma I, \tag{2.1}$$
$$\frac{dR}{dt} = \gamma I.$$

We observe that since the transmission rate $\lambda$ is positive, in the biologically relevant regime where the populations are positive, the susceptible population is strictly decreasing with respect to $t$ due to the first equation in (2.1). This indicates that $S(t)$ is a one-to-one function, and hence one can define an inverse function of $S$ and express $t$ in terms of $S$. Therefore, we can regard $I$ as a function of $S$, instead of $t$. By dividing the second equation in (2.1) by the first one, we obtain

$$\frac{dI}{dS} = \frac{\lambda IS - \gamma I}{-\lambda IS}, \tag{2.2}$$

from which we get

$$\frac{dI}{dS} = -1 + \frac{\gamma}{\lambda S}. \tag{2.3}$$

By solving (2.3) for $I$, we have

$$I(S) = -S + \frac{\gamma}{\lambda} \ln S + C, \tag{2.4}$$

where

$$C = I_0 + S_0 - \frac{\gamma}{\lambda} \ln S_0.$$

Now, by plugging (2.4) into the first equation in (2.1), we have

$$\frac{dS}{dt} = \lambda \left( S^2 - \frac{\gamma}{\lambda} S \ln S - CS \right). \tag{2.5}$$

Denote the anti-derivative of

$$\frac{1}{\lambda S^2 - \gamma S \ln S - \lambda CS}$$

by $F(S)$. Then we deduce from (2.5) that

$$F(S) - F(S_0) = t, \tag{2.6}$$

which is consistent with the analytical solution discovered by Harko, Lobo and Mak (2014). Once the susceptible population is computed from (2.6), the infected population is then calculated from (2.4), and $R$ is determined from the conservation of total population: $R = 1 - I - S$.

# 3   New Analysis for Classic SIR Model

Consider the equations of $S$ and $I$ in (2.1)

$$\begin{cases} \frac{d}{dt}S = -\lambda IS \\ \frac{d}{dt}I = \lambda IS - \gamma I, \\ S|_{t=0} = S_0, \ I|_{t=0} = I_0. \end{cases}$$

Note that $S_0 + I_0 \leq 1$ and $S_0 \geq 0$, $I_0 \geq 0$. Without loss of generality, we may assume $I_0 > 0$ since if $I_0 = 0$ then $(S_0, 0)$ is a steady state. Also we may assume $S_0 > 0$ since the case $S_0 = 0$ generates only the trivial dynamics $(S(t), I(t)) = (0, I_0 e^{-\gamma t})$. Now observe that since $I_0 > 0$, we have

$$0 < I(t) < 1, \qquad \forall \, 0 \leq t < \infty.$$

We now intend to *linearise* the nonlinear dynamics by introducing a new (nonlinear and nonlocal) time scale $\tau$ as

$$\tau(t) = \int_0^t I(s)ds, \qquad \text{i.e.} \quad d\tau = Idt.$$

We then introduce functions $(S_1, I_1)$ such that $S(t) = S_1(\tau(t))$, $I(t) = I_1(\tau(t))$. Then

$$\begin{cases} \frac{d}{d\tau}S_1 = -\lambda S_1, \\ \frac{d}{d\tau}I_1 = \lambda S_1 - \gamma, \\ S_1|_{\tau=0} = S_0, \ I_1|_{\tau=0} = I_0. \end{cases}$$

Solving the above system yields

$$\begin{cases} S_1(\tau) = S_0 e^{-\lambda \tau}, \\ I_1(\tau) = S_0 + I_0 - S_0 e^{-\lambda \tau} - \gamma \tau. \end{cases} \tag{3.1}$$

Observe that $I_1'' < 0$ and $I_1'(0) = \lambda S_0 - \gamma$. Now discuss two cases.

Case 1: $\lambda S_0 > \gamma$. In the literature this corresponds to the case where the reproduction number is bigger than 1. Clearly $I_1$ will first increase and then decrease to zero.
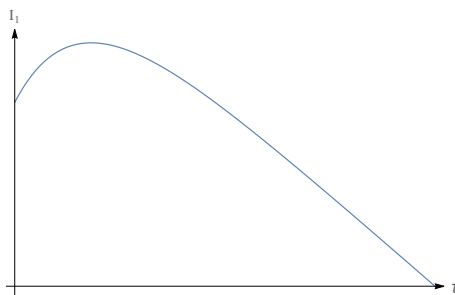


Figure 1: The function $I_1$ in Case 1

Case 2: $\lambda S_0 \leq \gamma$. This corresponds to the case where the reproduction number is less than or equal to 1. In this case $I_1$ will monotonically decrease to zero.
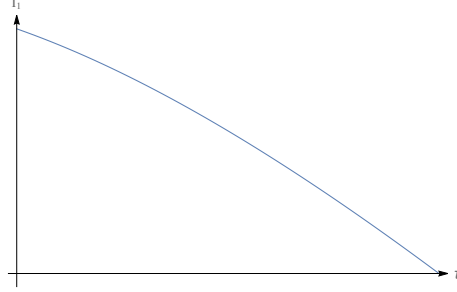


Figure 2: The function $I_1$ in Case 2

Concluding from the above two cases, it is clear that $I_1(\tau)$ is defined in the regime $0 \leq \tau \leq \tau_\infty$, where $\tau_\infty$ is the unique number solving the equation

$$S_0 + I_0 - S_0 e^{-\lambda \tau_\infty} - \gamma \tau_\infty = 0.$$

To recover $S(t) = S_1(\tau(t))$, one may first discretize the interval $[0, \tau_\infty]$ as $\tau_0, \tau_1, \cdots, \tau_m$, and solve

$$t_j = \int_0^{\tau_j} \frac{1}{I_1(s)} ds, \quad j = 0, \cdots, m.$$

Since $S(t_j) = S_1(\tau_j)$ and $I(t_j) = I_1(\tau_j)$, the dynamics of $(S, I)$ is then fully recovered. We should emphasize that through the rescaled time "clock" $\tau$, the dynamics for the original SIR model is fully captured by (3.1), and the map $\tau \to t$ simply provides the abscissa of the original SIR variables.

**Remark 3.1.** *Similar analysis can be conducted on the classic SIS (Hethcote 1976) epidemic model:*

$$\begin{cases} \frac{dS}{dt} = -\frac{\lambda}{N} SI + (\mu + \gamma)I, \\ \frac{dI}{dt} = \frac{\lambda}{N} SI - (\mu + \gamma)I, \end{cases}$$

*where $\lambda > 0$, $\gamma > 0$, $\mu \geq 0$ correspond to total transmission rate, average recovery rate and average death rate, respectively. Due to the conservation of $S(t) + I(t) \equiv N$, it suffices to study only the dynamics of $S$. In terms of the rescaled variable $\tilde{S} = S/N$ and the rescaled "clock" $d\tau = \tilde{I}dt$ ($\tilde{I} = I/N$), one has*

$$\frac{d\tilde{S}}{d\tau} = -\lambda \tilde{S} + \mu + \gamma$$

*which yields*

$$\tilde{S}(\tau) = \frac{\mu + \gamma}{\lambda}(1 - e^{-\lambda \tau}) + \tilde{S}_0 e^{-\lambda \tau},$$

$$\tilde{I}(\tau) = 1 - \frac{\mu + \gamma}{\lambda} + e^{-\lambda \tau}(\frac{\mu + \gamma}{\lambda} - \tilde{S}_0).$$

*One can then determine $\tau$ by using the relation $d\tau = \tilde{I}dt$. We omit the details. We should remark that alternatively one can regard $\tilde{S}$ as a function of $\tilde{I}$ and solve the ODE directly.*

6

# 4 Nonlocal SIR Model

In this section, we use the idea in Section 2 to derive analytical solutions to the (normalized) nonlocal differential-integral SIR model:

$$\frac{dS}{dt} = -\lambda IS,$$
$$I(t) = \int_0^t [1 - G(t - \tau)] \cdot [-S'(\tau)]d\tau + I_0[1 - G(t)], \tag{4.1}$$
$$R = 1 - I - S,$$

which becomes the classic normalized SIR model when $G(t) = 1 - e^{-\gamma t}$. Note that the integral equation for $I$ can be written as

$$I(t) = S_0 - S(t) + \int_0^t G(t - \tau)S'(\tau)d\tau + I_0[1 - G(t)]$$
$$= 1 - S(t) + \int_0^t G(t - \tau)S'(\tau)d\tau - I_0G(t) \quad \text{by recalling } R_0 = 0. \tag{4.2}$$

By regarding $t$ as a function of $S$, we have an alternative expression for $I$:

$$I(S) = 1 - S + \int_{S_0}^S \tilde{\mathbb{G}}(S, \mathbb{S})d\mathbb{S} - I_0\mathbb{G}(S), \tag{4.3}$$

which is a nonlocal equation with respect to $S$. Hence, one can postulate various conditions on $\tilde{\mathbb{G}}$ to generate different kinds of SIR models, depending on the specific biological environments under consideration. To proceed we consider the following *convolution*-type kernel

$$\tilde{\mathbb{G}}(S, \mathbb{S}) = \mathbb{G}(S - \mathbb{S}),$$

and (4.3) becomes (after a change of variable)

$$I(S) = 1 - S + \int_0^{S-S_0} \mathbb{G}(\mathbb{S})d\mathbb{S} - I_0\mathbb{G}(S), \tag{4.4}$$

Differentiating (4.4) with respect to $S$, we obtain

$$\frac{dI}{dS} = -1 + \mathbb{G}(S - S_0) - I_0\mathbb{G}'(S). \tag{4.5}$$

In view of (2.3) we see that when the kernel $\mathbb{G}$ satisfies

$$\mathbb{G}(S - S_0) - I_0\mathbb{G}'(S) = \frac{\gamma}{\lambda S}, \tag{4.6}$$

the following hybrid model

$$\frac{dS}{dt} = -\lambda IS,$$
$$I(S) = 1 - S + \int_{S_0}^S \mathbb{G}(S - \mathbb{S})d\mathbb{S} - I_0\mathbb{G}(S), \tag{4.7}$$
$$R = 1 - I - S$$

coincides with the classic normalized SIR model (2.1). By plugging the second equation of (4.7) into the first equation, we have

$$\frac{dS}{dt} = -\lambda S \left( 1 - S + \int_{S_0}^{S} \mathbb{G}(S - \mathbb{S}) d\mathbb{S} - I_0 \mathbb{G}(S) \right), \tag{4.8}$$

from which one gets

$$\frac{1}{\lambda S \left( 1 - S + \int_{S_0}^{S} \mathbb{G}(S - \mathbb{S}) d\mathbb{S} - I_0 \mathbb{G}(S) \right)} dS = -dt, \tag{4.9}$$

whose solution can be found depending on the specific form of $\mathbb{G}$ that one postulates.

## 5   An Approximation of the Nonlocal SIR Model

In this section, we use a similar approach as in Section 3 to derive new approximating solutions to the hybrid differential-integral SIR model in (4.1). We shall first derive an effective "$\tau$-model" by using a nonlocal stretching of the time axis. The properties of the $\tau$-model will then be analyzed in detail by some judicious choices of the interaction kernels.

### 5.1   Derivation of the $\tau$-Model

We introduce a new time scale $\tau$ as

$$\tau(t) = \int_0^t I(s) \, ds, \qquad \text{i.e.} \quad d\tau = I dt.$$

It is clear that for all practical situations $I(t)$ is positive as long as $S_0 \neq 0$ and $I_0 \neq 0$ (recall in the second equation of (4.1), one has $I(t) = \int_0^t (1 - G(t - \tilde{t}))(\lambda I(\tilde{t}) S(\tilde{t})) d\tilde{t} + I_0(1 - G(t))$. Hence, $\tau$ is a strictly increasing function, and thus, $\tau$ has an inverse function, which will be denoted as $\varphi$. Note that $\varphi(0) = 0$.

Let $S(t) = S_1(\tau(t))$, $I(t) = I_1(\tau(t))$. Then it follows from (4.1) that

$$S_1(\tau) = S_0 e^{-\lambda \tau};$$

$$I_1(\tau) = \lambda \int_0^\tau [1 - G(\varphi(\tau) - \varphi(\tilde{\tau}))] S_1(\tilde{\tau}) \, d\tilde{\tau} + I_0[1 - G(\varphi(\tau))].$$

Denote $G_1(\tau, \tilde{\tau}) = G(\varphi(\tau) - \varphi(\tilde{\tau}))$. Then we obtain

$$I_1(\tau) = \lambda S_0 \int_0^\tau [1 - G_1(\tau, \tilde{\tau})] e^{-\lambda \tilde{\tau}} \, d\tilde{\tau} + I_0[1 - G_1(\tau, 0)]$$

$$= \lambda S_0 e^{-\lambda \tau} \int_0^\tau [1 - G_1(\tau, \tau - \tilde{\tau})] e^{\lambda \tilde{\tau}} \, d\tilde{\tau} + I_0[1 - G_1(\tau, 0)],$$

where in the second equality above, we have made a change of variable $\tilde{\tau} \to \tau - \tilde{\tau}$. Since $G$ is a cumulative density function, the function $G_1(\tau, \tilde{\tau})$ vanishes when $\tau \leq \tilde{\tau}$, $G_1(\tau, \tilde{\tau})$ is monotonically increasing in the variable $\tau$ when $\tilde{\tau}$ is held fixed, and is monotonically decreasing

in the variable $\tilde{\tau}$ when $\tau$ is fixed, and $G_1(\tau, \tilde{\tau})$ tends to 1 as $\tau$ tends to infinity.[1] To proceed further we now dispose of the requirement that $G_1$ was obtained from a cumulative density function $G$ through a nonlinear map $\varphi$, and keep in stock only some (fairly mild) consistency conditions. As will become clear, such relaxation gives rise to a more general family of nonlocal models which includes the original ones as special cases. To this end, denote

$$D_- := \{(x, y) \in \mathbb{R}^2 : \ 0 \le y \le x < \infty\}.$$

The following definition seems quite natural.

**Definition 5.1** (Admissible kernels)**.** *A continuous function $G_1 = G_1(\tau, \tilde{\tau})$: $D_- \to [0, 1]$ is said to be admissible if the following conditions hold:*

- *$G_1(\tau, \tau) = 0$ for any $0 \le \tau < \infty$ and $G_1(\tau, \tilde{\tau}) \to 1$ as $\tau \to \infty$ for each fixed $\tilde{\tau}$.*

- *$G_1(\tau, \tilde{\tau})$ is non-decreasing in $\tau$ when $\tilde{\tau}$ is fixed, and $G_1(\tau, \tilde{\tau})$ is non-increasing in $\tilde{\tau}$ when $\tau$ is fixed.*

**Remark.** *The continuity requirement on $G_1$ is for convenience and simplicity only. It can certainly be weakened to partial continuity (in each variables separately) or upper or lower continuity depending on the convention used in the cumulative distribution function. We shall not dwell on this issue here.*

**Definition 5.2** ($\tau$-model)**.** *Let $\lambda > 0$. Let $G_1$ be a given admissible kernel, $S_0 > 0$, $I_0 > 0$. We say $(S_1, I_1)$ evolves according to a $\tau$-model with parameters $(\lambda, G_1)$ and initial data $(S_0, I_0)$ if $S_1(\tau) = e^{-\lambda \tau} S_0$, and*

$$I_1(\tau) = \lambda S_0 e^{-\lambda \tau} \int_0^\tau [1 - G_1(\tau, \tau - \tilde{\tau})] e^{\lambda \tilde{\tau}} \, \mathrm{d}\tilde{\tau} + I_0[1 - G_1(\tau, 0)], \quad \tau > 0.$$

**Remark.** *By using the discussion preceding the definition of admissible kernel, it is immediately clear that the nonlocal model (4.1) is a special case of our $\tau$-model through the identification $G_1(\tau, \tilde{\tau}) = G(\varphi(\tau) - \varphi(\tilde{\tau}))$. Here one tacitly assume that the model (4.1) is already solved on some time interval and there is no difficulty in defining the nonlinear (albeit not explicit) map $\varphi$. We ignore completely the subtle dependence of the map $\varphi$ on the cumulative distribution $G$ and treat the resulting $G_1$ function as given. To obtain more accurate models one can perform iterations on the $\tau$-model (by feeding iterated trial kernel functions $G_1^{(k)} = G(\varphi^{(k)}(\tau) - \varphi^{(k)}(\tilde{\tau}))$) which in some sense will correspond to performing fixed-point type arguments on the original nonlocal model.*

**Remark 5.3.** *Since $G_1(\tau, \tau - \tilde{\tau}) \to 0$ as $\tilde{\tau} \to 0$, we have $I_1(\tau) > 0$ for any $0 \le \tau < \infty$. On the other hand, by a change of variable*

$$I_1(\tau) = \lambda S_0 \int_0^\tau [1 - G_1(\tau, \tilde{\tau})] e^{-\lambda \tilde{\tau}} d\tilde{\tau} + I_0[1 - G_1(\tau, 0)].$$

---

[1]From a modeling point of view, one can regard $G_1(\tau, \tilde{\tau})$ as a new family of heterogeneous cumulative distribution functions coming from some prior probability distributions. For example, one can take a class of probability density kernels $p = p(\tau, \tau_0)$ which are defined on the region $\tau \ge \tau_0$ (think of $\tau_0$ as a parameter marking the time origin). Define $G_1(\tau, \tau_0) = \int_{\tau_0}^\tau p(s, \tau_0) ds$. Then clearly such function $G_1(\tau, \tau_0)$ represents a parameterized class of cumulative distributions which are heterogeneous in the sense that it has some dependence on the location $\tau_0$.

*By using Lebesgue Dominated Convergence, it follows that $I_1(\tau) \to 0$ as $\tau \to \infty$. It follows that the map*

$$\psi(\tau) = \int_0^\tau \frac{1}{I_1(\tilde{\tau})} d\tilde{\tau},$$

*provides a monotone and smooth bijection of the time axis $[0, \infty)$. Denote the inverse map of $\psi$ as $\psi^{-1}$. Then in the original time scale $t$, we can recover*

$$S(t) = S_1(\psi^{-1}(t)), \quad I(t) = I_1(\psi^{-1}(t)).$$

**Remark.** *In practice, the inverse map $\psi^{-1}$ can be recovered more efficiently using a direct discretization of the $\tau$-variables. More precisely one may (as before in the classic SIR case) first discretize the interval $[0, \infty)$ as $\tau_0, \tau_1, \tau_2, \cdots$, and solve*

$$t_j = \int_0^{\tau_j} \frac{1}{I_1(\tilde{\tau})} d\tilde{\tau}, \quad j = 0, 1, \cdots.$$

*Since $S(t_j) = S_1(\tau_j)$ and $I(t_j) = I_1(\tau_j)$, the dynamics of $(S, I)$ is then (approximately) fully recovered.*

Our $\tau$-model shares some general features in common with the classic models. As an example, the following proposition is a manifestation of the maximum principle in the nonlocal situation.

**Proposition 5.4** (Maximum principle)**.** *For all $\tau \geq 0$, $I_1(\tau) > 0$ and $I_1(\tau) \to 0$ as $\tau \to \infty$. Also for all $\tau > 0$,*

$$S_1(\tau) + I_1(\tau) \leq S_0 + I_0.$$

*Proof.* The properties of $I_1$ were proved in Remark 5.3. Now since $G_1 \geq 0$, it is clear that

$$S_1(\tau) + I_1(\tau) \leq \lambda S_0 e^{-\lambda \tau} \int_0^\tau e^{\lambda \tilde{\tau}} \, \mathrm{d}\tilde{\tau} + I_0$$
$$= S_0 + I_0 - S_0 e^{-\lambda \tau} \leq S_0 + I_0 \quad \text{for all } \tau.$$

$\square$

## 5.2 Simplified $\tau$-Models

Our $\tau$-model is quite flexible since different choices of the admissible kernel $G_1$ will lead to various new nonlocal SIR models. We now choose a homogeneous kernel $G_1(\tau, \tilde{\tau}) = \tilde{G}(\tau - \tilde{\tau})$, where $\tilde{G} : [0, \infty) \to [0, 1]$ is a given cumulative distribution function. Then clearly

$$I_1(\tau) = \lambda S_0 e^{-\lambda \tau} \int_0^\tau [1 - \tilde{G}(\tilde{\tau})] e^{\lambda \tilde{\tau}} \, \mathrm{d}\tilde{\tau} + I_0[1 - \tilde{G}(\tau)]. \tag{5.1}$$

The explicit form of $I_1$ can sometimes be worked out by using the specific form of $\tilde{G}$ that one postulates. Now we discuss a few simple cases which are analogous to the classic SIR model.

Case 1: $\tilde{G}(\tau) = 1 - e^{-A\tau}$ with $0 < A < \lambda$. It is easy to check that

$$I_1 = \frac{\lambda S_0}{A - \lambda} e^{-\lambda\tau} + (I_0 - \frac{\lambda S_0}{A - \lambda}) e^{-A\tau}.$$

Clearly $I_1$ will first increase and then decrease to zero as $\tau \to \infty$. See Figure 3. This one corresponds to the case where the reproduction number is bigger than 1 in the classic SIR model.
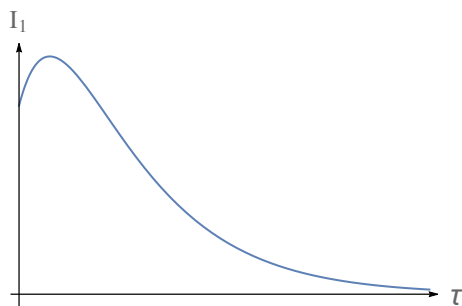


Figure 3: The function $I_1$ in Case 1

Case 2: $\tilde{G}(\tau) = 1 - e^{-A\tau}$ with $\lambda < A < \lambda \left(1 + \frac{S_0}{I_0}\right)$ and

$$I_1 = \frac{\lambda S_0}{A - \lambda} e^{-\lambda\tau} + (I_0 - \frac{\lambda S_0}{A - \lambda}) e^{-A\tau}.$$

In this case $I_1$ will monotonically decrease to zero as $\tau \to \infty$. See Figure 4. This one corresponds to the case where the reproduction number is less than or equal to 1 in the classical SIR model.
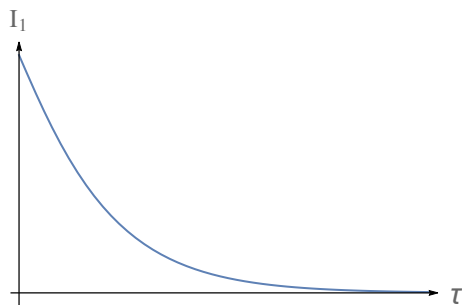


Figure 4: The function $I_1$ in Case 2

Case 3: An important difference from the classic SIR model is that in our $\tau$-model, $I_1$ may exhibit several bumps, that is, $I_1$ may have several local maximum points. To see this phenomenon in the case of at least two bumps, we shall need to make some further simplifications of the model (5.1) by yet another stretching of the time axis. Since this formulation will be needed for the remaining part of this section, we shall record it here as a proposition.

**Proposition 5.5.** *By using the change of variable $s = s(\tau) = e^{\lambda\tau}$, the model (5.1) is equivalent*

11

*to the integral equation:*

$$f(s) = \frac{\beta}{s} \int_1^s g(\tilde{s})\,\mathrm{d}\tilde{s} + g(s), \quad s \geq 1, \tag{5.2}$$

*where* $f(s) = \frac{I_1(\log(s)/\lambda)}{I_0}$, $g(s) = 1 - \tilde{G}(\log(s)/\lambda)$ *and* $\beta = S_0/I_0 > 0$. *Note that* $g(s)$ *is a given continuous non-increasing function defined on* $[1, \infty)$ *such that* $g(1) = 1$ *and* $g(s) \to 0$ *as* $s \to \infty$.

*Proof.* The proof is obvious. One should note that (for later discussions) the monotonicity of $I_1(\tau)$ is the same as the one of $f(s)$. Also observe that thanks to the transformation $s = e^{\lambda \tau}$, exponential decay in the $\tau$ variable then translates to the power decay in the $s$-variable. □

**Remark.** *From the relation*

$$f(s) = \beta \cdot \frac{1}{s} \int_1^s g(\tilde{s})\,\mathrm{d}\tilde{s} + g(s),$$

*one sees that* $f(s) \gtrsim s^{-1}$ *for all* $s$, *thus* $f$ *cannot have better than* $s^{-1}$ *decay.*

**Remark.** *As was already mentioned, our later results of this section will be stated in terms of the pair* $(g, f)$ *with* $g$ *being a given kernel function and* $f$ *being the infected population in "s-variable". The main difficulty in our constructions later is the requirement that the kernel function* $g(s)$ *must monotonically decrease to zero when* $s$ *tends to infinity. Somewhat surprisingly despite this rigid requirement there can still appear oscillations in the infected population by judicious choice of the kernel functions.*

**Remark 5.6.** *The map* $g \to f$ *in (5.2) is invertible and in this sense* $g$ *and* $f$ *is in one to one correspondence. To see this, one can rewrite (5.2) as*

$$\left(s^\beta \int_1^s g(\tilde{s})d\tilde{s}\right)' = f(s)s^\beta.$$

*Integrating and some simple algebra then yield*

$$g(s) = -\beta s^{-\beta-1} \int_1^s f(\tilde{s})\tilde{s}^\beta d\tilde{s} + f(s),$$

$$g'(s) = \left(s^{-\beta} \int_1^s \tilde{s}^\beta f(\tilde{s})d\tilde{s}\right)''$$

$$= \beta(\beta+1)s^{-\beta-2} \int_1^s f(\tilde{s})\tilde{s}^\beta d\tilde{s} - \beta\frac{f(s)}{s} + f'(s).$$

To see the phenomenon of having at least two peaks in $I_1$, we now give an explicit computable (albeit not smooth) example in terms of $f$. Choose $\beta = 2$ and a piecewise continuous function $g$ as:

$$g(s) := \begin{cases} \frac{1}{s}, & 1 \leq s \leq 2; \\ \frac{1}{2} & 2 \leq s \leq 2.1; \\ \frac{2.1}{2s} & s \geq 2.1. \end{cases}$$

12

Then
$$f(s) := \begin{cases} \frac{2\ln(s)+1}{s}, & 1 \le s \le 2; \\ \frac{3}{2} + \frac{2\ln 2 - 2}{s} & 2 \le s \le 2.1; \\ \frac{2}{s}\left(\ln 2 + \frac{1}{20} + \frac{2.1}{2}\ln(\frac{s}{2.1})\right) + \frac{2.1}{2s} & s \ge 2.1. \end{cases}$$

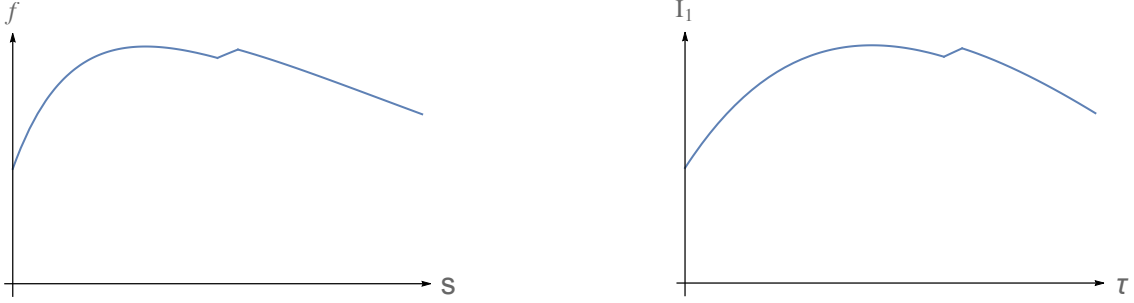The picture of this $f(s)$ and $I_1(\tau)$ can be illustrated in Figure 5 below:



Figure 5: The functions $f$, and $I_1$ (with $\lambda = 0.1$), in Case 3

Although the above $g$ and $f$ are not smooth, it provides some good evidence that one can hope to construct smooth solutions with two or more peaks. One should note, however, that a direct mollification of the $g$ function may introduce $O(1)$-changes for the derivative of $f$ near its original peak and thus may destroy the peak. On the other hand, if one prefers to mollifying directly the function $f$ near its two peaks, then one has to make sure that $g$ still remains monotonically decreasing after mollification, that is,

$$g'(s) = \beta(\beta+1)s^{-\beta-2}\int_1^s f(\tilde{s})\tilde{s}^\beta d\tilde{s} - \beta\frac{f(s)}{s} + f'(s) \le 0, \quad \forall s.$$

Now note that near the second peak of $f$, $f'$ is not continuous and it has $O(1)$-jump around the peak. Since $g'(2.1-) = 0$, the above inequality may well stop being true after mollification due to this $O(1)$-change. For these considerations we shall present in detail the construction and also the mollification procedure in the remaining part of this section. Our main results in the next subsection are Proposition 5.8 and Proposition 5.12 which give the existence of solutions having multiple peaks. To clarify the meaning of peak solution, we now introduce a simple definition.

**Definition 5.7** (Peak). *A point $s_0 \in (1, \infty)$ is said to be a peak (i.e. strict local max) for the function $f : [1, \infty) \to (0, \infty)$, if there exists a local neighborhood $\mathcal{N}_0$ of $s_0$, such that $f(s) < f(s_0)$ for any $s \in \mathcal{N}_0$, $s \ne s_0$. In particular, if $f$ is $C^2$, $f'(s_0) = 0$ and $f''(s_0) < 0$, then $s_0$ is a peak for $f$.*

Our main objective is to find kernel functions $g$ which decrease monotonically to zero such that the resulting infected population $f$ possesses multiple peaks. Thanks to the algebra in Remark 5.6, it suffices for us to construct a positive function $f$ with multiple peaks and obeying the nonlocal inequality $g' < 0$ (along with decay conditions at infinity) i.e.:

$$\beta(\beta+1)s^{-\beta-2}\int_1^s f(\tilde{s})\tilde{s}^\beta d\tilde{s} - \beta\frac{f(s)}{s} + f'(s) < 0, \quad \forall s \ge 1.$$

13

Achieving the above inequality point-wisely together with $C^\infty$-smoothness turns out to be a delicate task, and to elucidate the main ideas we shall present a rough result first (Proposition 5.8) which will be upgraded later in Proposition 5.12. In both propositions we shall slightly change the notation and denote $f$ by $I$, that is,

$$I(s) = \frac{\beta}{s} \int_1^s g(\tilde{s}) d\tilde{s} + g(s), \quad s \geq 1;$$

and equivalently

$$g(s) = -\beta s^{-\beta-1} \int_1^s I(\tilde{s}) \tilde{s}^\beta d\tilde{s} + I(s),$$

$$g'(s) = (s^{-\beta} \int_1^s \tilde{s}^\beta I(\tilde{s}) d\tilde{s})''$$

$$= \beta(\beta+1) s^{-\beta-2} \int_1^s I(\tilde{s}) \tilde{s}^\beta d\tilde{s} - \beta \frac{I(s)}{s} + I'(s).$$

## 5.3 Peak Solutions

Case 4a: existence of arbitrarily (finitely) many peaks. We first show that it is possible to admit finitely many peaks (as many as possible). A stronger result will be established in forthcoming Case 4b (compare with Proposition 5.12).

**Proposition 5.8** (Finitely many peaks, rough control). *Let $\beta > 0$. For any integer $m_0 \geq 1$ and any $0 < \theta \leq 1$, there exists a family of $C^\infty$-smooth functions $I = I(s) : [1, \infty) \to (0, \infty)$ such that*

$$\left( s^{-\beta} \int_1^s I(\tilde{s}) \tilde{s}^\beta d\tilde{s} \right)''$$

$$= \beta(\beta+1) s^{-\beta-2} \int_1^s I(\tilde{s}) \tilde{s}^\beta d\tilde{s} - \beta s^{-1} I(s) + I'(s) < 0, \quad \forall 1 \leq s < \infty. \tag{5.3}$$

*Furthermore $I(s) \lesssim s^{-\theta}$, $|I'(s)| \lesssim s^{-1-\theta}$, and $I(s)$ has at least $m_0$ strict local maxima located at some points $s_1, s_2, \cdots, s_{m_0}$, such that $I'(s_j) = 0$, $I''(s_j) < 0$ for any $1 \leq j \leq m_0$.*

**Remark 5.9.** *First observe that it suffices for us to construct a piecewise smooth $I(s)$ (overall as a Lipschitz continuous function) which satisfies (5.3) except at one point $s = \overline{S}_0$. Then mollifying such $I(s)$ around $s = \overline{S}_0$ suitably easily yields the desired smooth function still satisfying (5.3). Indeed, suppose $I(s)$ satisfies (5.3) for $1 \leq s < \overline{S}_0$ and $s > \overline{S}_0$ respectively, $I(s)$ decays to zero as $s$ tends to infinity and $I(s)$ has $m_0$ local maxima inside the interval $(1, \overline{S}_0)$. Also assume $I(\cdot)$ is $C^\infty$ for $s < \overline{S}_0$ and $s > \overline{S}_0$ respectively, and the LHS of (5.3) is bounded above by some uniform negative constant for $s$ close to $\overline{S}_0$. Then*

$$g(s) = \left( s^{-\beta} \int_1^s I(\tilde{s}) \tilde{s}^\beta d\tilde{s} \right)' = -\beta s^{-\beta-1} \int_1^s I(\tilde{s}) \tilde{s}^\beta d\tilde{s} + I(s)$$

*satisfies $g'(s) < 0$ for $s < \overline{S}_0$ and $s > \overline{S}_0$, and $g'(s) < -c_0 < 0$ for $s$ close to $\overline{S}_0$ ($c_0 > 0$ is some constant). This (together with the decay of $g$ at infinity) implies that $g$ is Lipschitz, $g > 0$ for all $s$*

*and g is strictly monotonically decreasing. Define $g_{\text{smooth}}$ to be a suitable mollification of g (the mollification is only done in a sufficiently small neighborhood of $s = \overline{S}_0$). Define*

$$I_{\text{smooth}}(s) = s^{-\beta} \cdot \left(s^{\beta} \int_1^s g_{\text{smooth}}(\tilde{s}) d\tilde{s}\right)' = \beta \frac{\int_1^s g_{\text{smooth}}(\tilde{s}) d\tilde{s}}{s} + g_{\text{smooth}}(s).$$

*Then $I_{\text{smooth}}$ is clearly positive, $C^{\infty}$ and satisfies (5.3). Furthermore $I_{\text{smooth}}$ has at least $m_0$ local maxima since it inherits the local maxima of $I(\cdot)$ inside the interval $(1, \overline{S}_0)$ as long as the mollification is done away from the local maxima (note that $I_{\text{smooth}}$ may generate some new maxima near the point $\overline{S}_0$ since mollification is done for the g function).*

**Remark 5.10.** *We clarify a bit more in detail the "suitable" mollification procedure for the g-function alluded to in the previous remark. The main point is to preserve strict monotonicity whilst achieving $C^{\infty}$-smoothness. To be precise, assume g is $C^{\infty}$ except at $s = \overline{S}_0$, $g' < 0$ for $s \neq \overline{S}_0$, and $g'(s) < -c_0 < 0$ for $|s - \overline{S}_0| < 2\delta_0$ ($\delta_0 > 0$ is sufficiently small). Choose a $C^{\infty}$-smooth radial decreasing bump function $\phi_0$ such that $\phi_0(x) = \exp(-\frac{1}{1-x^2})$ for $\frac{2}{3} < |x| < 1$, $\phi_0(x) = 1$ for $|x| < \frac{1}{3}$, and $\phi_0(x) = 0$ for $|x| \geq 1$. Define*

$$h(s) = g'(s)\left(1 - \phi_0(\frac{s - \overline{S}_0}{\delta_0})\right) - a \cdot \phi_0(\frac{s - \overline{S}_0}{\delta_0}),$$

*where the constant $a > 0$ is chosen such that*

$$\int_{\overline{S}_0 - \delta_0}^{\overline{S}_0 + \delta_0} h(s) ds = \int_{\overline{S}_0 - \delta_0}^{\overline{S}_0 + \delta_0} g'(s) ds.$$

*Define $c_1 = \min\{c_0, a\} > 0$. Then clearly for all $|s - \overline{S}_0| \leq \delta_0$, one has*

$$h(s) < -c_1 < 0.$$

*Also $h(s) = g'(s)$ for $|s - \overline{S}_0| > \delta_0$. Define*

$$g_{\text{smooth}}(s) = g(1) + \int_1^s h(\tilde{s}) d\tilde{s}.$$

*Then it is not difficult to check that $g_{\text{smooth}}(s) = g(s)$ for $s < \overline{S}_0 - \delta_0$ or $s > \overline{S}_0 + \delta_0$, and $g'_{\text{smooth}} < 0$ for all s. Thus $g_{\text{smooth}}$ is the desired $C^{\infty}$ function.*

**Remark.** *The basic idea of the construction of the function $I(s)$ is as follows. First if we ignore the constraint of having $m_0$ local maxima, we can choose $I(s)$ piece-wisely as*

$$I(s) = \begin{cases} 1, & 1 \leq s < \overline{S}_0, \\ (\frac{\overline{S}_0}{s})^{\theta}, & s > \overline{S}_0, \end{cases}$$

*where $0 < \theta \leq 1$. It is easy to check that the inequality (5.3) holds for $s < \overline{S}_0$. For $s > \overline{S}_0$, one observes that*

$$s^{-\beta} \int_1^s I(\tilde{s}) \tilde{s}^{\beta} d\tilde{s} = s^{-\beta} \int_1^{\overline{S}_0} \tilde{s}^{\beta} d\tilde{s} + s^{-\beta} \int_{\overline{S}_0}^s \tilde{s}^{\beta} \cdot (\frac{\overline{S}_0}{\tilde{s}})^{\theta} d\tilde{s}$$

$$= s^{-\beta}\left(\frac{\overline{S}_0^{\beta+1} - 1}{\beta + 1} - \frac{\overline{S}_0^{\beta+1}}{\beta - \theta + 1}\right) + s^{1-\theta}\frac{\overline{S}_0^{\theta}}{\beta - \theta + 1}.$$

15

*This then clearly yields (5.3) thanks to the condition $0 < \theta \le 1$. Now to finish the construction we can use an "$\epsilon$-perturbation" idea to create $m_0$ local maxima in the range $1 \le s < \overline{S}_0$. The mollification then shows that one has at least $m_0$ local maxima.*

*Proof.* Define

$$I(s) = \begin{cases} 1 + \epsilon \cdot e^{-s} \sin s, & 1 \le s \le \overline{S}_0; \\ \alpha_1(\frac{\overline{S}_0}{s})^\theta, & s > \overline{S}_0, \end{cases}$$

where $\alpha_1 = 1 + \epsilon e^{-\overline{S}_0} \sin \overline{S}_0$, and the parameters $(\epsilon, \overline{S}_0)$ will be specified momentarily. It will become clear from the description below that we obtain a family of functions. Furthermore $I(s)$ is piecewise $C^\infty$ and by using mollification (see Remarks 5.9 and 5.10) $I(s)$ can be made arbitrarily smooth.

First it is easy to check that the function $e^{-s} \sin s$ has strict local maxima at the points $s = (n+\frac{1}{4})\pi$, $n = 0, 2, 4, \cdots$. We can take $\overline{S}_0 = (2(m_0-1)+\frac{1}{4}+\eta_0)\pi$ with $\eta_0 > 0$ being sufficiently small. This obviously guarantees that $I(\cdot)$ admits precisely $m_0$ strict local maxima located inside the interval $(1, \overline{S}_0)$.

Next observe that in (5.3) if we plug in the main order $I(s) \equiv 1$ we obtain the value $-\beta$. Thus by choosing $0 < \epsilon < 1$ sufficiently small ($\epsilon$ can depend on $\overline{S}_0$ and $\beta$) one can guarantee that the LHS of (5.3) is always less than $-\beta/2$ for $1 \le s < \overline{S}_0$.

Finally we need to check (5.3) for $\overline{S}_0 < s < \infty$. Clearly

$$s^{-\beta} \int_1^s I(\tilde{s})\tilde{s}^\beta d\tilde{s} = s^{-\beta} \int_1^{\overline{S}_0} \tilde{s}^\beta d\tilde{s} + s^{-\beta} \int_1^{\overline{S}_0} \tilde{s}^\beta \cdot \epsilon e^{-\tilde{s}} \sin \tilde{s} d\tilde{s} + \alpha_1 s^{-\beta} \int_{\overline{S}_0}^s \tilde{s}^\beta \cdot (\frac{\overline{S}_0}{\tilde{s}})^\theta d\tilde{s}$$

$$= s^{-\beta}\left(\frac{\overline{S}_0^{\beta+1}-1}{\beta+1} - \alpha_1 \frac{\overline{S}_0^{\beta+1}}{\beta-\theta+1} + C_1 \cdot \epsilon\right) + \alpha_1 s^{1-\theta} \frac{\overline{S}_0^\theta}{\beta-\theta+1},$$

where $C_1 = \int_1^{\overline{S}_0} \tilde{s}^\beta e^{-\tilde{s}} \sin \tilde{s} d\tilde{s}$. By choosing $\epsilon < \frac{1}{2(\beta+1)} \cdot \frac{1}{|C_1|+1}$, one can clearly fulfill (5.3) for $\overline{S}_0 < s < \infty$. Furthermore the LHS of (5.3) is bounded above by some uniform negative constant for $s$ close to $\overline{S}_0$.

$\square$

Case 4b: More refined results: existence of arbitrarily finitely many peaks with precise control of the location of maxima.

**Lemma 5.11.** *Let $\beta > 0$ and $1 < \overline{S}_0 < \infty$. There exists $\theta_0 = \theta_0(\beta, \overline{S}_0) \in (0, 1)$, such that for any $0 < \theta < \theta_0$, there exists $f : [1, \infty) \to (0, \infty)$ satisfying the following properties:*

- *$f \in C^\infty$, and $|f(s)| \lesssim s^{-\theta}$, $|f'(s)| \lesssim s^{-\theta-1}$ for all $s$.*

- *For some $0 < \eta_0 < 1$ ($\eta_0 > 0$ is a tunable parameter which can be made as small as possible), one has $f(s) \equiv 1$ for $1 \le s \le \overline{S}_0 - \eta_0$, and $f'(s) < 0$ for all $s > \overline{S}_0 - \eta_0$.*

- *There exists a constant $c_1 > 0$, such that for all $1 \le s < \infty$,*

$$\left(s^{-\beta} \int_1^s f(\tilde{s})\tilde{s}^\beta d\tilde{s}\right)'' + c_1 s^{-\theta-1} < 0. \tag{5.4}$$

*Proof.* We first define a $C^1$-function as:

$$f_0(\tau) = \begin{cases} 1, & 1 \le \tau \le \overline{S}_0, \\ \frac{1}{1-\theta}((\frac{\overline{S}_0}{\tau})^\theta - \theta\frac{\overline{S}_0}{\tau}), & \tau > \overline{S}_0, \end{cases}$$

where $\theta \in (0,1)$ will be taken sufficiently small. For (5.4) the regime $s \le \overline{S}_0$ is easy to check. For $s > \overline{S}_0$, we have

$$s^{-\beta} \int_1^s f_0(\tilde{s})\tilde{s}^\beta d\tilde{s}$$

$$= s^{-\beta}\left(\frac{\overline{S}_0^{\beta+1} - 1}{\beta + 1} - \frac{1}{1-\theta} \cdot \frac{1}{\beta - \theta + 1}\overline{S}_0^{\beta+1} + \frac{\theta}{1-\theta} \cdot \frac{1}{\beta}\overline{S}_0^{\beta+1}\right)$$

$$+ \frac{1}{1-\theta} \cdot \frac{1}{\beta - \theta + 1} \cdot \overline{S}_0^\theta \cdot s^{1-\theta} - \frac{\theta}{1-\theta} \cdot \overline{S}_0 \cdot \frac{1}{\beta}.$$

Now note that if we take $\theta$ such that

$$\frac{\theta}{1-\theta} \cdot \frac{1}{\beta}\overline{S}_0^{\beta+1} \le \frac{1}{\beta + 1},$$

then we obtain

$$s^{-\beta} \int_1^s f_0(\tilde{s})\tilde{s}^\beta d\tilde{s}$$

$$= -A_1 s^{-\beta} + A_2 s^{1-\theta} - A_3,$$

where $A_i > 0$ are constants. This clearly yields a concave function satisfying (5.4).

Now to finish the construction we just need to mollify $f_0$ around $\overline{S}_0$ whilst keeping the constraint (5.4) (with a smaller constant $c_1$ if necessary). Let $\phi_0 \in C_c^\infty(\mathbb{R})$ be radial decreasing such that $\phi_0(x) = \exp(-\frac{1}{1-x^2})$ for $\frac{2}{3} < |x| < 1$, $\phi_0(x) = 1$ for $|x| < \frac{1}{3}$, and $\phi_0(x) = 0$ for $|x| \ge 1$.

Take small constant $\eta_0 > 0$ and define

$$h(s) = f_0'(s)\left(1 - \phi_0(\frac{s - \overline{S}_0}{\eta_0})\right) - a\phi_0(\frac{s - \overline{S}_0}{\eta_0}),$$

where $a > 0$ satisfies

$$a \int \phi_0(x)dx = -\int f_0'(\overline{S}_0 + \eta_0 x)\phi_0(x)dx.$$

Observe that $h(s) < 0$ for all $s > \overline{S}_0 - \eta_0$. Define $f(s) = f_0(1) + \int_1^s h(\tilde{s})d\tilde{s}$. Clearly $f(s) = f_0(s)$, $f'(s) = f_0'(s)$ for $|s - \overline{S}_0| > \eta_0$. For $|s - \overline{S}_0| \le \eta_0$, we have $|f'(s) - f_0'(s)| = O(\eta_0)$, $|f(s) - f_0(s)| = O(\eta_0^2)$. Then for $|s - \overline{S}_0| < \eta_0$, we have

$$\left(s^{-\beta} \int_1^s f(\tilde{s})\tilde{s}^\beta d\tilde{s}\right)''$$

$$= \beta(\beta + 1)s^{-\beta-2}\int_1^s f(\tilde{s})\tilde{s}^\beta d\tilde{s} - \beta s^{-1}f(s) + f'(s)$$

$$= \beta(\beta + 1)s^{-\beta-2}\int_1^s f_0(\tilde{s})\tilde{s}^\beta d\tilde{s} - \beta s^{-1}f_0(s) + f_0'(s) + O(\eta_0).$$

By further shrinking $\eta_0$ it is then easy to fulfill the constraint (5.4). $\qquad\square$

17

**Remark.** *Alternatively, one can use the usual convolution to define (a slightly different version of ) $f(s)$ as follows: first extend $f_0(s)$ to $\mathbb{R}$ such that $f_0(s) = 1$ for $s \leq 0$. Then for $\eta_0 > 0$ let $\phi_{\eta_0}(x) = \eta_0^{-1}\phi_0(x/\eta_0)/\|\phi_0\|_1$. Define $f$ as the restriction of the convolution $f_0 * \phi_{\eta_0}$ to the axis $[1, \infty)$. Note that $f \equiv 1$ on $[1, \overline{S}_0 - \eta_0]$ and $f' < 0$ on $(\overline{S}_0 - \eta_0, \infty)$. The estimates*

$$|f(s) - f_0(s)| \lesssim s^{-1-\theta} \cdot \eta_0, \quad \forall\, s,$$
$$|f'(s) - f_0'(s)| \lesssim s^{-2-\theta} \cdot \eta_0, \quad \forall\, s > \overline{S}_0 + \eta_0,$$
$$|f'(s)| \lesssim \eta_0, \quad \forall\, \overline{S}_0 - \eta_0 \leq s \leq \overline{S}_0 + \eta_0,$$

*then yield the result.*

The following is a strengthened version of Proposition 5.8.

**Proposition 5.12** (Finitely many peaks, precise control). *Let $\beta > 0$. For any $1 < M_1 < M_2 < \infty$, any integer $m_0 \geq 1$, and any $m_0$ different points $\{s_j\}_{j=1}^{m_0}$ (WLOG $s_1 < s_2 < \cdots < s_{m_0}$) in the interval $(M_1, M_2)$, one can find a $C^\infty$-smooth function $I = I(s) : [1, \infty) \to (0, \infty)$ such that the following hold:*

- *For all $1 \leq s < \infty$,*

$$\left( s^{-\beta} \int_1^s I(\tilde{s})\tilde{s}^\beta d\tilde{s} \right)''$$
$$= \beta(\beta + 1)s^{-\beta-2}\int_1^s I(\tilde{s})\tilde{s}^\beta d\tilde{s} - \beta s^{-1}I(s) + I'(s) < 0. \tag{5.5}$$

- *For some $0 < \theta < 1$, $I(s) \lesssim s^{-\theta}$, $|I'(s)| \lesssim s^{-1-\theta}$, for all $s \geq 1$.*

- *$I(s)$ has $m_0$ strict local maxima located at the points $s_1, s_2, \cdots, s_{m_0}$, such that $I'(s_j) = 0$, $I''(s_j) < 0$ for any $1 \leq j \leq m_0$.*

- *$I(s)$ has no other peaks besides the points $\{s_j\}_{j=1}^{m_0}$. More precisely, there are points $a_1 < b_1 < a_2 < b_2 < \cdots < a_{m_0} < b_{m_0} < a_{m_0+1}$, such that*

  - *$I(s) = 1$ for $1 \leq s \leq a_1$, $b_j \leq s \leq a_{j+1}$ $(j = 1, \cdots, m_0)$;*
  - *$I(a_{m_0+1}) = 1$ and $I'(s) < 0$ on $(a_{m_0+1}, \infty)$;*
  - *On each interval $(a_j, b_j)$, $I(s)$ has a peak at $s = s_j$, and $I'(s) > 0$ for $a_j < s < s_j$, $I'(s) < 0$ for $s_j < s < b_j$.*

*Proof.* We first let $\overline{S}_0 = M_2$ and choose $f(s)$ by using Lemma 5.11. Take a radial decreasing bump function[2] $\phi_1 \in C_c^\infty(\mathbb{R})$ with $\phi_1(x) = e^{-\frac{1}{1-x^2}}$ for $|x| < 1$, $\phi_1(x) = 0$ for $|x| > 1$. Define

$$I(s) = f(s) + \sum_{j=1}^{m_0} \delta_0^3 \phi_1\left(\frac{s - s_j}{\delta_0}\right),$$

where $\delta_0 \ll \min_{0 \leq j \leq m_0}|s_{j+1} - s_j|$ (here $s_0 = M_1$, $s_{m_0+1} = M_2 - \eta_0$ where $\eta_0$ is in Lemma 5.11) will be taken sufficiently small (such that (5.5) is satisfied). For $1 \leq j \leq m_0$, define $a_j = s_j - \delta_0$, $b_j = s_j + \delta_0$. Define $a_{m_0+1} = M_2 - \eta_0$. It is then not difficult to check that $I(s)$ fulfills the desired properties. $\square$

---

[2]Note that here we choose the bump function $\phi_1$ to have a "strict" peak instead of a flat maximum at the center.

**Remark 5.13.** *It is even possible to modify slightly the above construction and construct a function $I(s)$ having infinitely many peaks within $(M_1, M_2)$. Take a sequence of increasing points $s_j \to s_\infty \in (M_1, M_2)$. Define*

$$I(s) = f(s) + \sum_{j=1}^{\infty} c_j \phi_1 \left( \frac{s - s_j}{\delta_j} \right),$$

*where $0 < \delta_j \ll \min\{s_{j+1} - s_j, s_j - s_{j-1}, 2^{-j}\}$ for all $j$ (so that bubbles stay disjoint) and $c_j = e^{-2/\delta_j}$ (one may also need to shrink $\delta_j$ further so that (5.5) is satisfied). Easy to check that for any integer $k \geq 0$,*

$$\sum_j c_j \delta_j^{-k} \lesssim \sum_j k! \cdot e^{-\frac{1}{\delta_j}} < \infty.$$

*Thus $I$ is $C^\infty$-smooth which has infinitely many (small) peaks located at the points $s_j$.*

**Remark.** *Another natural idea is to look for a function $I(s)$ possessing infinitely many plateau segments emanating to infinity (from which one can create infinitely many peaks in each plateau such that the formed peaks will have positive distances from each other). However one can dis-prove such a possibility. More precisely, recall that*

$$I(s) = \beta \cdot \frac{1}{s} \int_1^s g(\tilde{s}) d\tilde{s} + g(s), \quad \forall s \geq 1,$$

*and $g > 0$ monotonically decrease to zero as $s$ tends to infinity.*

*Claim: The function $I(s)$ cannot have infinitely many intervals $[a_n, b_n]$ with $a_n, b_n \to \infty$ such that $I(s) \equiv c_n$ for some constant $c_n$ on the interval $[a_n, b_n]$.*

*Proof of Claim.* Argue by contradiction. Suppose such $I(s)$ exists. Then on each $[a_n, b_n]$, we have

$$\left( s^\beta \int_1^s g(\tilde{s}) d\tilde{s} \right)' = c_n s^\beta.$$

Solving this on $[a_n, b_n]$ then gives

$$\int_1^s g(\tilde{s}) d\tilde{s} = \frac{c_n}{\beta + 1} s - d_n s^{-\beta}, \quad s \in [a_n, b_n];$$

$$g(s) = \frac{c_n}{\beta + 1} + \beta d_n s^{-\beta - 1}, \quad s \in [a_n, b_n],$$

where $d_n \geq 0$ (since $g$ is non-increasing). It follows that

$$\frac{1}{s} \int_1^s g(\tilde{s}) d\tilde{s} \leq g(s), \quad \text{for } s = a_n, n \geq 1.$$

This is impossible since there exists some number $\alpha_0 > 0$, such that

$$\frac{1}{s} \int_1^s g(\tilde{s}) d\tilde{s} > g(s), \quad \forall s > \alpha_0. \tag{5.6}$$

This last inequality can be easily proved by an $\epsilon$-$\delta$ argument using the monotonicity of $g$ and the fact that $g$ tends to zero as $s \to \infty$. $\qquad\square$

19

**Remark.** *Inspired by the preceding remark, one can prove a stronger result which asserts that the function $I(s)$ must be strictly monotonically decreasing near infinity. Thus the construction in Remark 5.13 is in some sense the best possible since one cannot have peaks propagating to infinity.*

*Claim: There exists $\alpha_0 > 0$, such that $I'(s) < 0$ for all $s > \alpha_0$.*

*Proof of Claim.* This follows directly from the identity

$$I'(s) = g'(s) + \frac{\beta}{s}\left(-\frac{1}{s}\int_1^s g(\tilde{s})d\tilde{s} + g(s)\right).$$

and the inequality (5.6). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

**Remark.** *From the point of view of epidemic modeling, one should perhaps not expect infinitely many (relatively) large outbreaks which is naturally excluded in our nonlocal model.*

# 6 Concluding Remarks

In this work we analyzed the classic SIR model and one of its recently developed nonlocal variations. The latter is a hybrid differential-integral model incorporating general probability transition kernels as parameters. We discussed two approaches. One is based on quantifying the explicit relation of the infected population versus the susceptible population. The basic observation is that the susceptible population strictly decreases in time which makes it a natural time arrow. In the classic SIR model it is possible to give almost explicit and analytic expressions of the infected population in terms of the susceptible population. In the nonlocal SIR model one can use an approximation of the kernel to derive the explicit functional relationship between the infected population and the susceptible population. In our second approach, we introduced a nonlocal time parameter whose differential is the reciprocal of the infected population. Under this nonlocal stretching of the time axis, the classic SIR model becomes completely linear and the dynamics can be easily classified. The functional relation between the new time arrow and the old time can be explicitly worked out by using the classification result obtained in the first step. Exploiting a similar idea we analyzed the nonlocal SIR model and discovered several novel peak solutions.

Compared with the classic SIR model an interesting new feature of the nonlocal SIR model is the appearance of solutions having multiple peaks. From a practical point of view, we expect that the new feature can help understand certain epidemiological situations which can hardly be explained by the classic SIR model. For instance, as one of the major epidemic outbreaks, a continuing source outbreak usually exhibits multiple peaks during the epidemic period. An example of this type of outbreak is cholera from a contaminated water supply, which is caused because multiple people are exposed at differing times from the same source of disease (drinking water), so the epidemic incidences spike at different times. For such kind of epidemiological situation, one can collect historical data for (suitably normalized infected population) $I(s)$, and reconstruct the approximate kernel $g(s)$. In this way one can build a nonlocal model to capture the trend of the historical data and make future predictions of possible outbreaks.

On the other hand, from the analytical point of view the stability/instability of the peak solutions as well as the classification of general asymptotics of such solutions are presently not known

and these seem to be interesting new directions. Another fundamental issue left unaddressed in this work is the notion of the reproduction number in general nonlocal epidemic models. Identifying the correct generalization of the reproduction number as well as building the natural connection between the nonlocal and the classic epidemic models seem to be a challenging task. We hope to address some of these issues in future works.

## Acknowledgements

## References

[1] Batiha AM, Batiha B (2011) A new method for solving epidemic model. Australian Journal of Basic and Applied Sciences 5:3122-3126.

[2] Biazar J (2006) Solution of the epidemic model by Adomian decomposition method. Applied Mathematics and Computation 173:1101-1106.

[3] Hethcote HW (1976) Qualitative analyses of communicable disease models. Mathematical Biosciences 28:335-356.

[4] Harko T, Lobo FSN, Mak MK (2014) Exact analytical solutions of the Susceptible-Infected-Recovered (SIR) epidemic model and of the SIR model with equal death and birth rates. Applied Mathematics and Computation 236:184-194.

[5] Kermack WO, McKendrick AG (1927) Contribution to the mathematical theory of epidemics. Proceedings of the Royal Society of London A 115:700-721.

[6] Korobeinikov A, Wake GC (2002) Lyapunov functions and global stability for SIR, SIRS, and SIS epidemiological models. Applied Mathematics Letters 15:955-960.

[7] Li H, Peng R, Wang Z (2018) On a diffusive Susceptible-Infected-Susceptible epidemic model with mass action mechanism and birth-death effect: analysis, simulations, and comparison with other mechanisms. SIAM Journal on Applied Mathematics 78:2129-2153.

[8] Lou Y, Zhao X (2010) The periodic Ross-Macdonald model with diffusion and advection. Applicable Analysis 89:1067-1089.

[9] Lou Y, Zhao X (2011) A reaction-diffusion malaria model with incubation period in the vector population. Journal of Mathematical Biology 62:543-568.

[10] O'Regan SM, Kelly TC, Korobeinikov A, O'Callaghan MJA, Pokrovskii AV (2010) Lyapunov functions for SIR and SIRS epidemic models. Applied Mathematics Letters 23:446-448.

[11] Rafei M, Daniali H, Ganji DD (2007) Variational iteration method for solving the epidemic model and the prey and predator problem. Applied Mathematics and Computation 186:1701-1709.

[12] Rafei M, Ganji DD, Daniali H (2007) Solution of the epidemic model by homotopy perturbation method. Applied Mathematics and Computation 187:1056-1062.

[13] Williams HT, Mazilu I, Mazilu DA (2012) Stochastic epidemic-type model with enhanced connectivity: exact solution. Journal of Statistical Mechanics: Theory and Experiment P01017.

L. Guan
Department of Mathematics, Tulane University
New Orleans, LA 70118, USA

Email: lguan@tulane.edu

D. Li
Department of Mathematics, The Hong Kong University of Science and Technology
Clear Water Bay, Kowloon, Hong Kong

Email: madli@ust.hk

K. Wang
Department of Mathematics, The Hong Kong University of Science and Technology
Clear Water Bay, Kowloon, Hong Kong

Email: kewang@ust.hk

K. Zhao
Department of Mathematics, Tulane University
New Orleans, LA 70118, USA

Email: kzhao@tulane.edu